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A simple derivation and interpretation of the third integral in Stellar Dynamics

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ABSTRACT

Starting from the problem of two fixed centres we find a simple derivation of its third integral in terms of the scalar product of the angular momenta about the two fixed centres. This is then generalised to find the general form of the potential in which an exact third integral exists.

Key words: Stellar Dynamics, Galaxies, Star–Clusters.

1 INTRODUCTION

When deriving or using the third integral of the motion, i.e., that other than energy or angular momentum about the axis, I have often been asked for its physical meaning. The new derivation given here may not completely answer that question, but the realisation that the kinetic part is the dot product of the angular momenta about the two centres, or two foci, goes somewhat further towards an answer than anything I have seen previously.

This paper gives what I hope is a simpler, cleaner derivation of the integral, in elementary terms that do not require spheroidal coordinates or the separation of the Hamilton–Jacobi equation. Earlier treatments requiring such sophistications are found in Lynden-Bell (1962), Eddington (1915), de Zeeuw (1985abc), Stackel (1890) and Kuzmin (1956). The two dimensional problem is discussed in Whittaker (1904). Contopoulos (1960) showed that in many galactic potentials of non separable form, a regular third integral exists for most orbits.

2 PRELIMINARIES - THE ONE CENTRE PROBLEM

Hamilton's beautiful treatment of this problem is not widely enough taught (Hamilton, 1847). As we need some of the steps later we give his treatment here. The equation of motion is

$$\ddot{\mathbf{r}} = -\mu\hat{\mathbf{r}}/r^2 \ , \tag{1}$$

where $\mu = Gm$ and $\hat{\mathbf{r}}$ is the unit vector. Cross multiplying by $\mathbf{r}$ and writing $\mathbf{h} = \mathbf{r}\times\dot{\mathbf{r}}$ we find

$$\dot{\mathbf{h}} = \mathbf{r}\times\ddot{\mathbf{r}} = 0 \ ,$$

so $\mathbf{h}$ is constant. Cross multiplying (1) by $\mathbf{h}$

$$d\left(\mathbf{h}\times\dot{\mathbf{r}}\right)/dt = -\mu\left(\hat{\mathbf{r}}\times\dot{\mathbf{r}}/r\right)\times\hat{\mathbf{r}} = -\mu d\hat{\mathbf{r}}/dt \ , \tag{2}$$

the last identity may readily be demonstrated by dull algebra but the physical understanding is more interesting. In a small interval δt, $\hat{\mathbf{r}}\times\delta\mathbf{r}/r$ is perpendicular to the movement $\delta\mathbf{r}$ and to $\hat{\mathbf{r}}$ and gives the angle moved about the centre. Hence $\hat{\mathbf{r}}\times\dot{\mathbf{r}}/r$ is the angular velocity of the radius vector which we call $\boldsymbol{\omega}$. Hence $(\hat{\mathbf{r}}\times\dot{\mathbf{r}}/r)\times\hat{\mathbf{r}} = \boldsymbol{\omega}\times\hat{\mathbf{r}}$ where $\boldsymbol{\omega}$ is the angular velocity of the unit vector $\hat{\mathbf{r}}$. However, the change in a unit vector is just $\boldsymbol{\omega}\times\hat{\mathbf{r}}$ because it only changes by rotation since its length is fixed so $d\hat{\mathbf{r}}/dt = \boldsymbol{\omega}\times\hat{\mathbf{r}}$.

Equation (2) is readily integrated to give

$$\dot{\mathbf{r}}\times\mathbf{h} = \mu(\hat{\mathbf{r}}+\mathbf{e}) \ , \tag{3}$$

where $\mathbf{e}$ is a constant of integration. Since both $\hat{\mathbf{r}}$ and $\hat{\mathbf{r}}\times\mathbf{h}$ are perpendicular to $\mathbf{h}$ we find $\mathbf{e}\cdot\mathbf{h} = 0$. Take the scalar product of (3) with $\hat{\mathbf{r}}/\mu$ then, on use of

$$\hat{\mathbf{r}}\cdot(\dot{\mathbf{r}}\times\mathbf{h}) = (\mathbf{r}\times\dot{\mathbf{r}})\cdot\mathbf{h}/r = h^2/r$$

we find

$$\ell/r = 1 + \mathbf{e}\cdot\hat{\mathbf{r}} \ , \tag{4}$$

where $\ell = h^2/\mu$. This is the equation of a general conic where $\mathbf{e}\cdot\hat{\mathbf{r}} = e\cos\phi$ with ϕ measured from pericentre and e the eccentricity. So we now identify Hamilton's vector $\mathbf{e}$ as the vector that points to the pericentre and whose length is the eccentricity. We note in passing the useful expression for the velocity found from (3),

$$\dot{\mathbf{r}} = \mu h^{-2}\mathbf{h}\times(\hat{\mathbf{r}}+\mathbf{e}) \ . \tag{5}$$

Its square may be used to derive the energy integral

3 THE THIRD INTEGRAL FOR THE TWO CENTRE PROBLEM

We measure $\mathbf{r_1}$ from the first centre of attraction at $-\mathbf{a}$ and $\mathbf{r_2}$ from the second at $+\mathbf{a}$ so $\mathbf{r_1} = \mathbf{r_2} + 2\mathbf{a}$. The equation of motion is

$$\ddot{\mathbf{r_1}} = \ddot{\mathbf{r}}_2 = -\mu_1 \hat{\mathbf{r}}_1/r_1^2 - \mu_2 \hat{\mathbf{r}}_2/r_2^2 \ . \qquad (6)$$

The energy and the component of angular momentum along the line of centres are, of course, conserved. We are interested in an independent third integral. Cross multiplying (6) by $\mathbf{r_1}$ we find

$$\dot{\mathbf{h}}_1 = d\left(\mathbf{r}_1 \times \dot{\mathbf{r_1}}\right)/dt = -\mu_2 \mathbf{r}_1 \times \hat{\mathbf{r}}_2/r_2^2 = -\mu_2 2\mathbf{a} \times \hat{\mathbf{r}}_2/r_2^2 \ ,$$

taking the scalar product with $\mathbf{h}_2$ we find *c.f.* (2),

$$\mathbf{h}_2 \cdot \dot{\mathbf{h}}_1 = -\mu_2 2\mathbf{a} \cdot \left[\hat{\mathbf{r}}_2 \times \left(\hat{\mathbf{r}}_2 \times \dot{\mathbf{r}}_2/r_2\right)\right] = \mu_2 2\mathbf{a} \cdot d\hat{\mathbf{r}}_2/dt \ .$$

Adding a half of this to the conjugate expression with $\mathbf{r}_2$ and $\mathbf{r}_1$ exchanged and $-\mathbf{a}$ for $\mathbf{a}$ we find

$$d\left(\tfrac{1}{2}\mathbf{h}_1 \cdot \mathbf{h}_2\right)/dt = d\left[\mathbf{a} \cdot \left(\mu_2 \hat{\mathbf{r}}_2 - \mu_1 \hat{\mathbf{r}}_1\right)\right]/dt \ . \qquad (7)$$

So

$$I_3 = \tfrac{1}{2}\mathbf{h}_1 \cdot \mathbf{h}_2 + \left(\mu_1 \hat{\mathbf{r}}_1 - \mu_2 \hat{\mathbf{r}}_2\right) \cdot \mathbf{a} = \text{constant} \ , \qquad (8)$$

which is the third integral for this problem. Using

$$\widetilde{r} = \tfrac{1}{2}(r_1 + r_2) \quad ; \quad a\widetilde{\mu} = (r_1 - r_2)/2$$

as coordinates the three integrals may now be used to solve for the orbits, but our purpose here is to generalise this beautifully simple derivation of (8) to more general problems. When $\mu_2 = 0$, $\mathbf{h}_2 = \mathbf{h}_1 - 2\mathbf{a} \times \dot{r}_1$ and the integral reduces to a combination of $\mathbf{h}_1$ and $\mathbf{e}$.

4 THE EXACT THIRD INTEGRAL MORE GENERALLY

We write our axially symmetrical potential ψ in the form $\psi(r_1,\ r_2)$. The equation of motion is

$$\ddot{\mathbf{r}}_1 = \ddot{\mathbf{r}}_2 = \boldsymbol{\nabla}\psi = \partial\psi/\partial r_1 \hat{\mathbf{r}}_1 + \partial\psi/\partial r_2 \hat{\mathbf{r}}_2 \ .$$

Performing the same steps as led to (7) we find in its place

$$d\left(\tfrac{1}{2}\mathbf{h}_1 \cdot \mathbf{h}_2\right)/dt = -r_2^2 \partial\psi/\partial r_2 \mathbf{a} \cdot d\hat{\mathbf{r}}_2/dt + r_1^2 d\psi/dr_1 \mathbf{a} \cdot d\hat{\mathbf{r}}_1/dt \ , \qquad (9)$$

which reduces to our former result when $\psi = \mu_1/r_1 + \mu_2/r_2$.

To proceed we need expressions for $\mathbf{a} \cdot \hat{\mathbf{r}}_2$, etc. These come from squaring the expression for $\mathbf{r}_1$ in terms of $\mathbf{a}$ and $\mathbf{r}_2$.

$$r_1^2 = r_2^2 + 4a^2 + 4\mathbf{a} \cdot \hat{\mathbf{r}}_2 r_2$$

so

$$r_2^2 d\left(\mathbf{a} \cdot \hat{r}_2\right)/dt = \tfrac{1}{2} r_1 r_2 \dot{r}_1 - \left[\tfrac{1}{4}\left(r_1^2 + r_2^2\right) - a^2\right]\dot{r}_2 \ . \qquad (10)$$

Collecting terms in $\dot{r}_1$ and in $\dot{r}_2$ in (9) we have using (10) and its conjugate with $r_1 \longleftrightarrow r_2$ and $a \longrightarrow -a$

$$d\left(\tfrac{1}{2}\mathbf{h}_1 \cdot \mathbf{h}_2\right)/dt = \left\{-\tfrac{1}{2} r_1 r_2 \partial\psi/\partial r_2\right) \qquad (11)$$
$$+ \left[\tfrac{1}{4}\left(r_1^2 + r_2^2\right) - a^2\right] d\psi/dr_1 \Big\} \dot{r}_1 + \text{conjugate} \ .$$

This last expression may be rewritten

$$\left\{ \begin{array}{l} -\frac{1}{2}\partial/\partial r_2 \left(r_1 r_2 \psi\right) + \\ \\ +\frac{1}{4}\partial/\partial r_1 \left[\left(r_1^2 + r_2^2 - 4a^2\right)\psi\right] \end{array} \right\} \dot{r}_1 + \text{conjugate} \ .$$

The second term in the curly bracket gives rise to an expression which, taken with its conjugate is a total time derivative.

We now ask for the condition on ψ that the remainder plus its conjugate be a perfect derivative. For this to be true it must be of the form $\partial\Lambda/\partial r_1\ \dot{r}_1 + \partial\Lambda/\partial r_2\ \dot{r}_2$ so $\partial/\partial r_2$ of the first term must be $\partial/\partial r_1$ of its conjugate.

Writing $\chi = r_1 r_2 \psi$ our condition reduces to

$$\partial^2\chi/\partial r_1^2 - \partial^2\chi/\partial r_2^2 = 0 \ ,$$

but this is the wave equation and its general solution is

$$\chi = \zeta(r_1 + r_2) - \eta(r_1 - r_2) \ ,$$

where ζ and η are arbitrary functions of those arguments. Thus the potential ψ must take the form

$$\psi = \left[\zeta(r_1 + r_2) - \eta(r_1 - r_2)\right]/(r_1 r_2) \ .$$

With this expression for ψ, (12) takes the form

$$\frac{d}{dt}\left(\tfrac{1}{2}\mathbf{h}_1 \cdot \mathbf{h}_2\right) =$$
$$= \frac{\partial}{\partial r_1}\left\{\left[\tfrac{1}{4}\left(r_1^2 + r_2^2\right) - a^2\right]\psi - \tfrac{1}{2}\left(\zeta + \eta\right)\right\}\dot{r}_1 +$$
$$+\text{conjugate},$$

so

$$\begin{aligned} I_3 &= \tfrac{1}{2}\mathbf{h}_1 \cdot \mathbf{h}_2 - \left[\tfrac{1}{4}\left(r_1^2 + r_2^2\right) - a^2\right]\frac{\zeta - \eta}{r_1 r_2} + \tfrac{1}{2}\left(\zeta + \eta\right) = \text{const} \\ &= \tfrac{1}{2}\mathbf{h}_1 \cdot \mathbf{h}_2 - \\ &\quad \frac{1}{r_1 r_2}\left\{\left[\left(\frac{r_1 - r_2}{2}\right)^2 - a^2\right]\zeta - \left[\left(\frac{r_1 + r_2}{2}\right)^2 - a^2\right]\eta\right\} \ . \end{aligned}$$

The final term reduces with suitable definitions to the $-(\mu\zeta - \lambda\eta)/(\lambda - \mu)$ form one finds in standard treatments (Lynden-Bell, 1962).

When the coordinates are oblate $\mathbf{a}$ is imaginary $i|\mathbf{a}|$ so $\mathbf{h}_1$ and $\mathbf{h}_2$ are angular momenta $(\mathbf{r} + i|\mathbf{a}|) \times \dot{\mathbf{r}}$, etc, about the imaginary points $\mathbf{r} = \pm i|\mathbf{a}|$. However the product of these angular momenta is still real. The expressions for the integral are unchanged except that when the r_1 and r_2 are interpreted as the greatest and least distances of a general point from the ring $z = 0,\ r = |a|$ then the $-a^2$ terms in the final expression of I_3 should be omitted.

I am unable to extend this derivation into the relativistic régime while maintaining its simplicity. Carter (1968) has shown that a third integral exists for a charged particle moving in the Kerr-Newman metric. When Newton's G is set equal to zero that metric leaves behind the most interesting electromagnetic field with a net charge q_1 (discussed in Lynden-Bell (2001)).

$$\mathbf{E} + i\mathbf{B} = -\nabla\left[q_1/\sqrt{(\mathbf{r} - i\mathbf{a}) \cdot (\mathbf{r} - i\mathbf{a})}\right] \quad ; \quad \mathbf{a} = (0, 0, a) \ .$$

Third integrals in this and other electromagnetic fields in the relativistic régime are related to the classical third integral discussed above. For a demonstration of this relationship and a derivation of the forms of electromagnetic fields in which they exist see Lynden-Bell (2000). Here we merely quote the form the third integral takes for the above electromagnetic field with the orbiting particle of mass m and charge q. $\hat{\boldsymbol{\phi}}$ is the unit toroidal vector, Φ and $\mathbf{A} = A\hat{\boldsymbol{\phi}}$ the electromagnetic scalar and vector potentials.

$$I = \tfrac{1}{2}\mathbf{h}_1 \cdot \mathbf{h}_2 + q\mu\zeta_3(\lambda)/(\lambda - \mu)$$

where

$$\begin{aligned}
\mathbf{h}_1 &= (\mathbf{r} - i\mathbf{a}) \times \mathbf{p} \;\; ; \;\; \mathbf{h}_2 = (\mathbf{r} + i\mathbf{a}) \times \mathbf{p} \\
\zeta_3(\lambda) &= q_1\sqrt{\lambda - a^2}\;\lambda^{-1}c^{-2}\left(\epsilon\lambda - \tfrac{1}{2}qq_1\sqrt{\lambda - a^2} - ach\right) \\
\mathbf{p} &= m\mathbf{v}/\sqrt{1 - V^2/c^2} + qA\hat{\boldsymbol{\phi}}/c \;\; ; \;\; \mathbf{v} = \dot{\mathbf{r}} \\
\epsilon &= mc^2/\sqrt{1 - V^2/c^2} + q\Phi \\
h &= mR^2\dot{\phi}/\sqrt{1 - V^2/c^2} + qRA/c \\
\Phi &= q_1\sqrt{\lambda - a^2}/(\lambda - \mu) \\
A/R &= a^2\lambda^{-1}\Phi \\
2\lambda &= r^2 + a^2 + |\,(\mathbf{r} - i\mathbf{a})^2\,| \\
&\quad \text{(constant on oblate confocal spheroids)} \\
2\mu &= r^2 + a^2 - |\,(\mathbf{r} - i\mathbf{a})^2\,| \\
&\quad \text{(constant on confocal hyperboloids of one sheet)}
\end{aligned}$$

I, h and ϵ are the integrals, $R^2 = x^2 + y^2 = \lambda\mu/a^2$, and the $|\;\;|$ sign is used in the sense of complex numbers.

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